D-Mercator: Network embedding into ultra low-dimensional hyperbolic spaces

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UNIVERSITAT DE BARCELONA

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UBICS

Why do we need the embeddings in higher dimensions?

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Mercator: uncovering faithful hyperbolic embeddings of complex networks

Guillermo García-Pérez[8,1,2], Antoine Allard[8,3,4], M Ángeles Serrano[5,6,7] and Marián Boguñá[5,6]

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Hyperbolic Graph Convolutional Neural Networks

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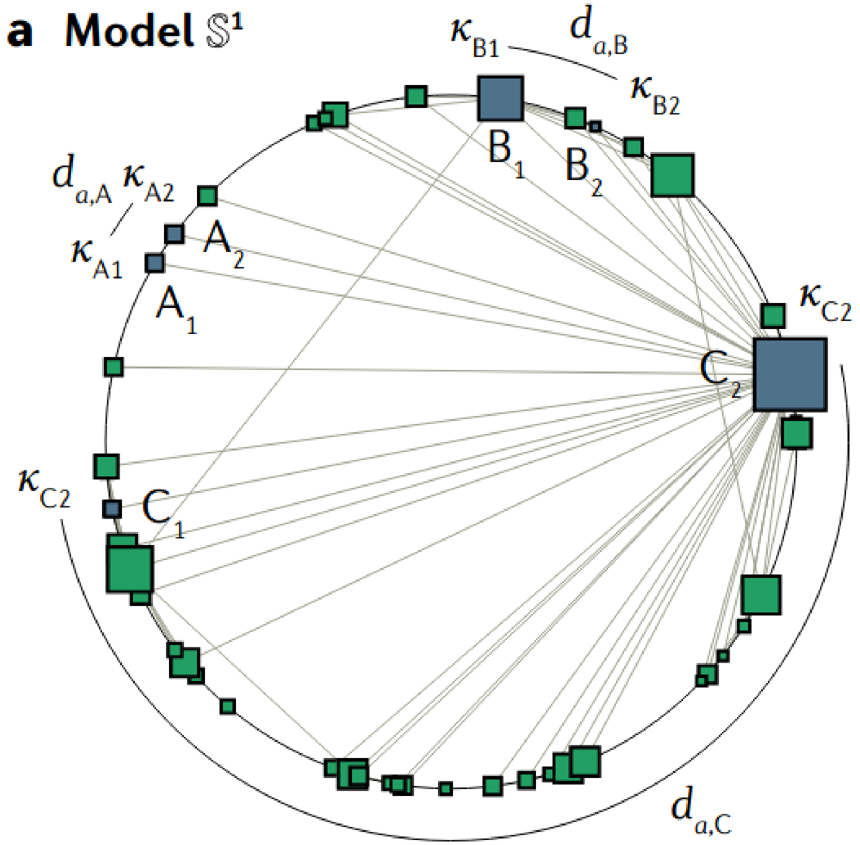
What is the „correct" hyperbolic dimension of real network?

$\mathbb{S}^1/\mathbb{H}^2$ model – overview

- Similarity space – circle of radius R ($N = 2\pi R$)
- Each node has a hidden degree κ
- Probability of connection

$$p_{ij} = \frac{1}{1 + \left(\frac{d_{ij}}{\mu\kappa_i\kappa_j}\right)^{\beta}}$$

- μ – controls average degree
- β – controls level of clustering
- $d_{ij} = R\Delta\theta_{ij}$



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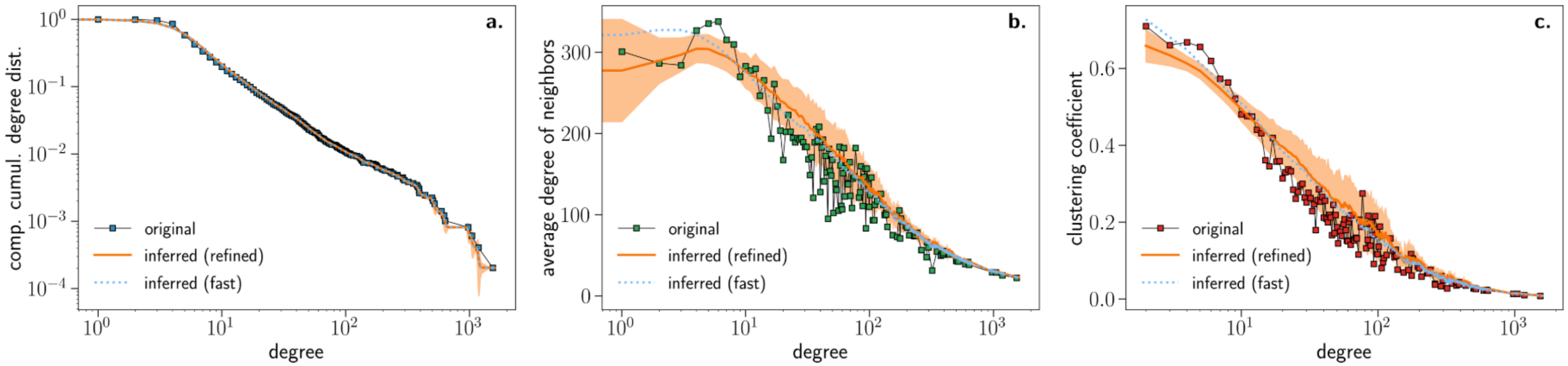
Boguñá, M., Bonamassa, I., De Domenico, M. *et al.* Network geometry. *Nat Rev Phys* **3**, 114–135 (2021). https://doi.org/10.1038/s42254-020-00264-4

Advantages of $\mathbb{S}^1/\mathbb{H}^2$ model

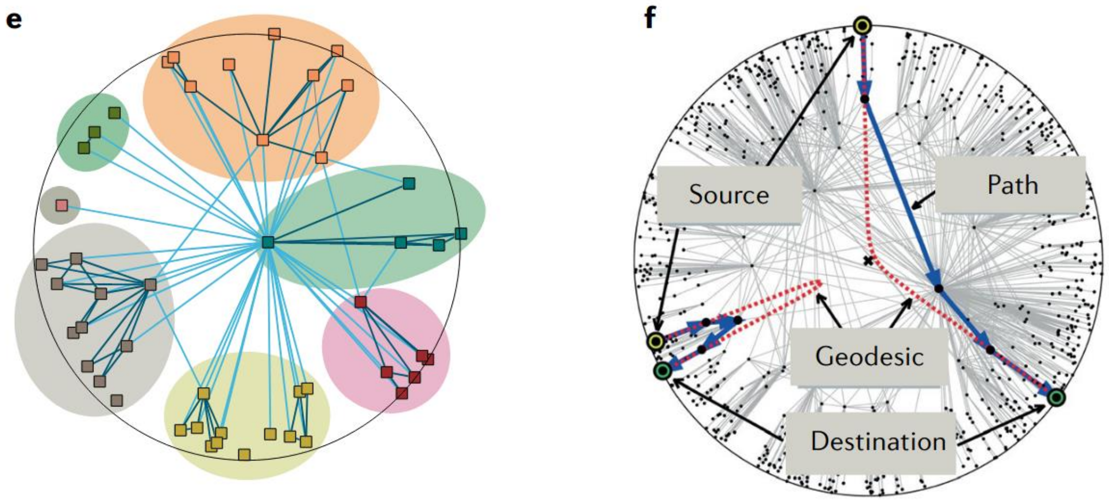
- Model explains typical properties of real networks
  - small-world property
  - heterogenous degree distribution
  - high level of clustering



García-Pérez, G., Allard, A., Serrano, M. Á., & Boguñá, M. (2019). Mercator: uncovering faithful hyperbolic embeddings of complex networks. *New Journal of Physics*, *21*(12), 123033.

Advantages of $\mathbb{S}^1/\mathbb{H}^2$model

- Embedding of real networks provides
  - efficient navigation
  - detection of communities
  - scale-down and scale-up network replicas



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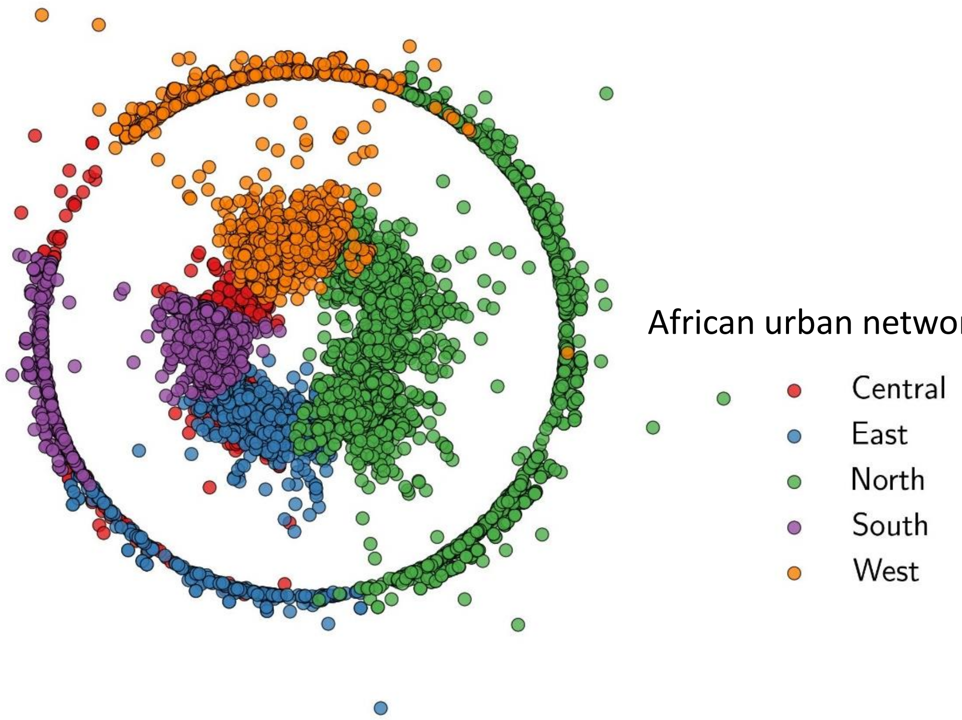
Limitations of $\mathbb{S}^1/\mathbb{H}^2$model

1. Overlapping communities



African urban network

Désy, B., Desrosiers, P., & Allard, A. (2022). Dimension matters when modeling network communities in hyperbolic spaces. *arXiv preprint arXiv:2209.09201*.
Almagro, P., Boguna, M., & Serrano, M. (2021). Detecting the ultra low dimensionality of real networks. *arXiv preprint arXiv:2110.14507*.

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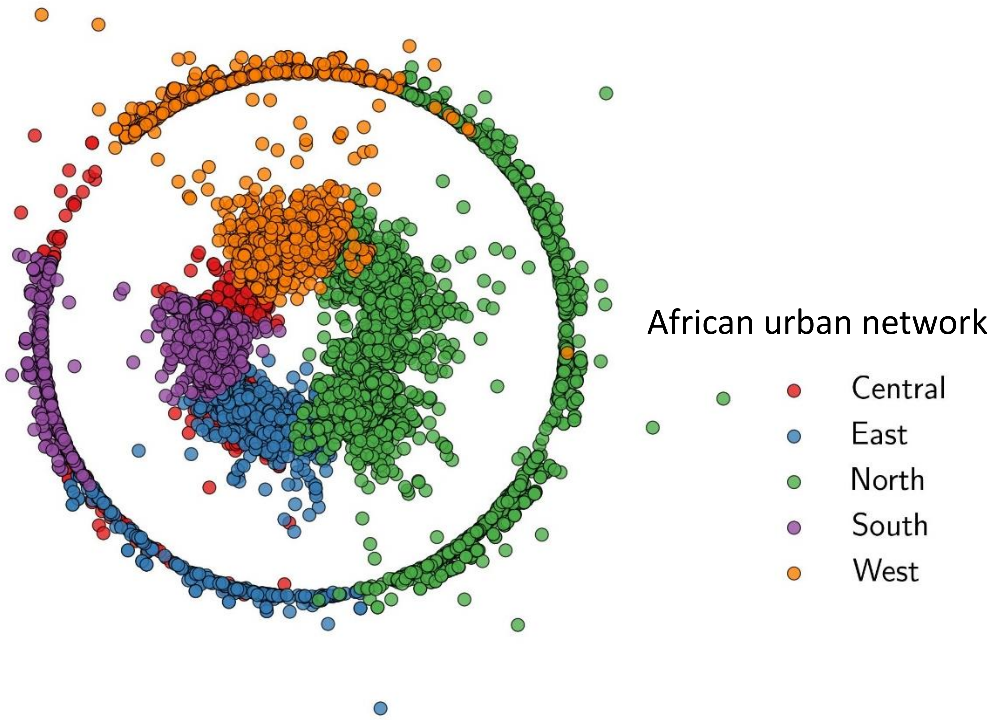
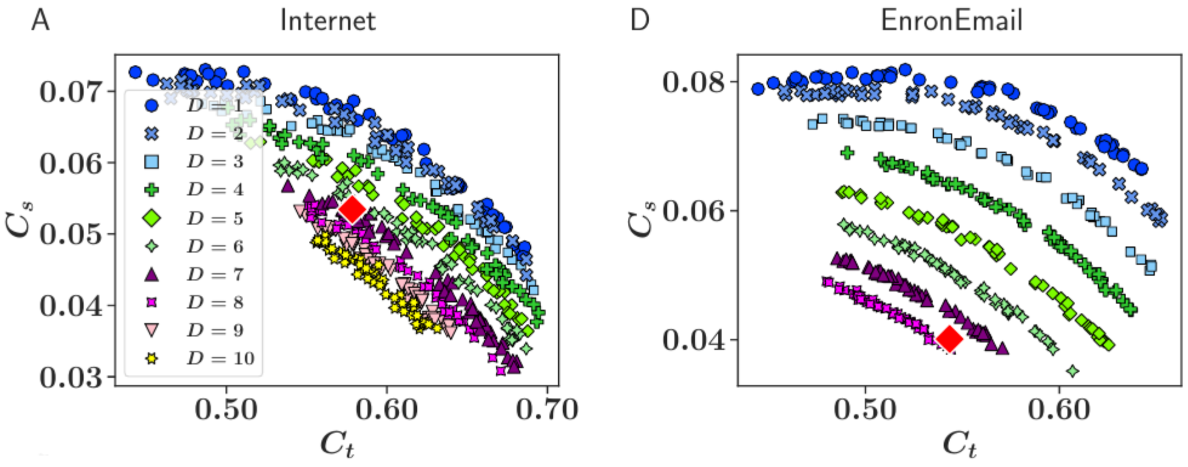


2. Estimated dimension based on cycles statistics is usually larger than 1



Mean densities of edge triangles (C_t) vs squares (C_s)

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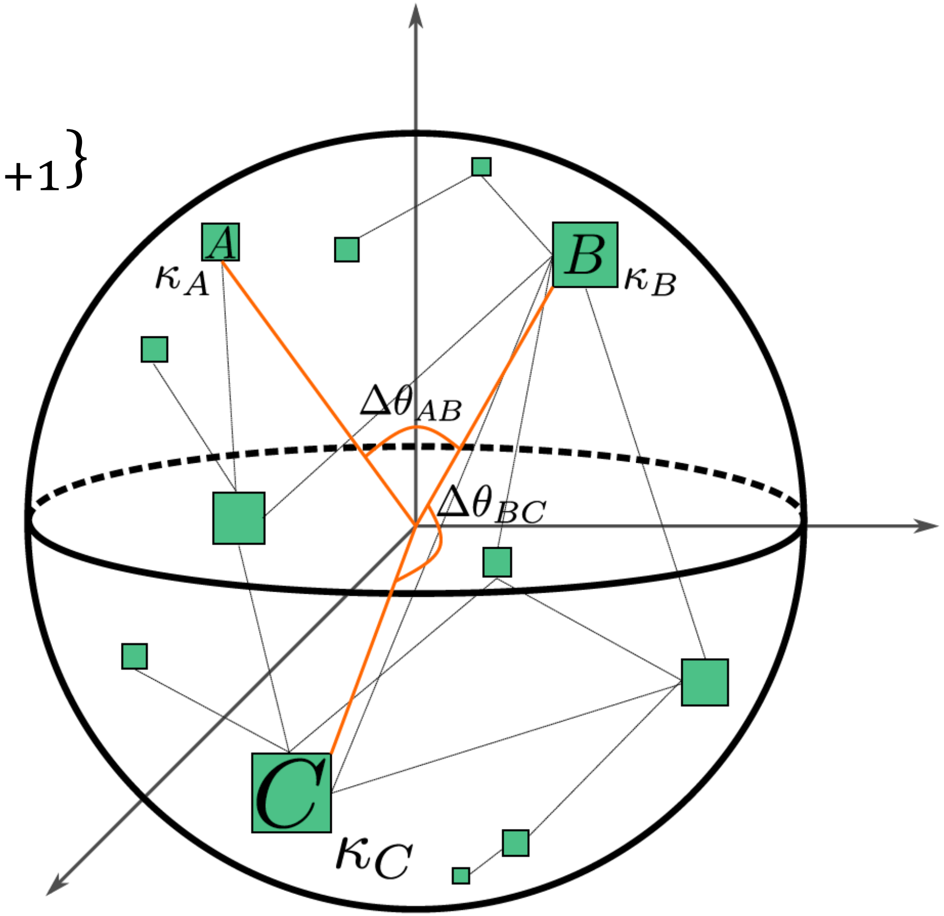
$\mathbb{S}^D/\mathbb{H}^{D+1}$ model

- A node i is endowed with a two variables
  - Hidden degree κ_i
  - Position in the similarity space $\mathbf{v_i} = \{\mathrm{v}_1, \mathrm{v}_2, \dots, \mathrm{v}_{\mathrm{D+1}}\}$
- The connection probability between node i and j

$$p_{ij} = \frac{1}{1 + \left(\frac{d_{ij}}{(\mu\kappa_i\kappa_j)^{\frac{1}{D}}}\right)^{\beta}}$$



- $d_{ij} = R\Delta\theta_{ij} = R \arccos\left(\frac{\mathbf{v}_i \cdot \mathbf{v_j}}{\|\mathbf{v}_i\|\|\mathbf{v}_j\|}\right)$

Serrano, M., Krioukov, D., & Boguñá, M. (2008). Self-Similarity of Complex Networks and Hidden Metric Spaces. *Phys. Rev. Lett., 100, 078701.*

D-Mercator:
Network embedding into
ultra low-dimensional hyperbolic spaces

Outline of the method

1. Inferring the hidden degrees
2. Inferring parameter β
3. Initial nodes' positions with Laplacian Eigenmaps
4. Final nodes' positions with Likelihood Maximization
5. Adjusting hidden degrees

Inferring hidden degrees and parameter β

1. Initialize hidden degrees κ-s as observed degrees in the real network
2. Compute the expected degree for each node according to the $\mathbb{S}^D$ model

$$\bar{k}(\kappa_i) = \frac{\Gamma\left(\frac{D+1}{2}\right)}{\sqrt{\pi}\Gamma\left(\frac{D}{2}\right)} \sum_{j \neq i} \int_0^{\pi} \frac{\sin^{D-1}\theta \, d\theta}{1 + \left(\frac{R\theta}{(\mu\kappa_i\kappa_j)^{\frac{1}{D}}}\right)^{\beta}}$$

3. Correct hidden degrees until $\max\left\{\left|\bar{k}(\kappa_i) - k_i\right|\right\} < \epsilon$
4. Initialize mean local clustering for each degree class
5. Compute expected mean local clustering spectrum
6. Accept given value of β if $|\bar{c} - \bar{c}^{emp}| < \epsilon_{\bar{c}}$, otherwise adjust β and recompute hidden degrees.

Inferring nodes' positions

- Laplacian Eigenmaps
  - Entries in weight matrix defined as

$$\omega_{ij} = e^{-\frac{|\mathbf{x}_i - \mathbf{x}_j|^2}{t}} = e^{-\frac{2\sin\langle\Delta\theta_{ij}\rangle}{t}}$$

t – scalling factor, $\langle\Delta\theta_{ij}\rangle$ – expected angular distance between nodes i and j

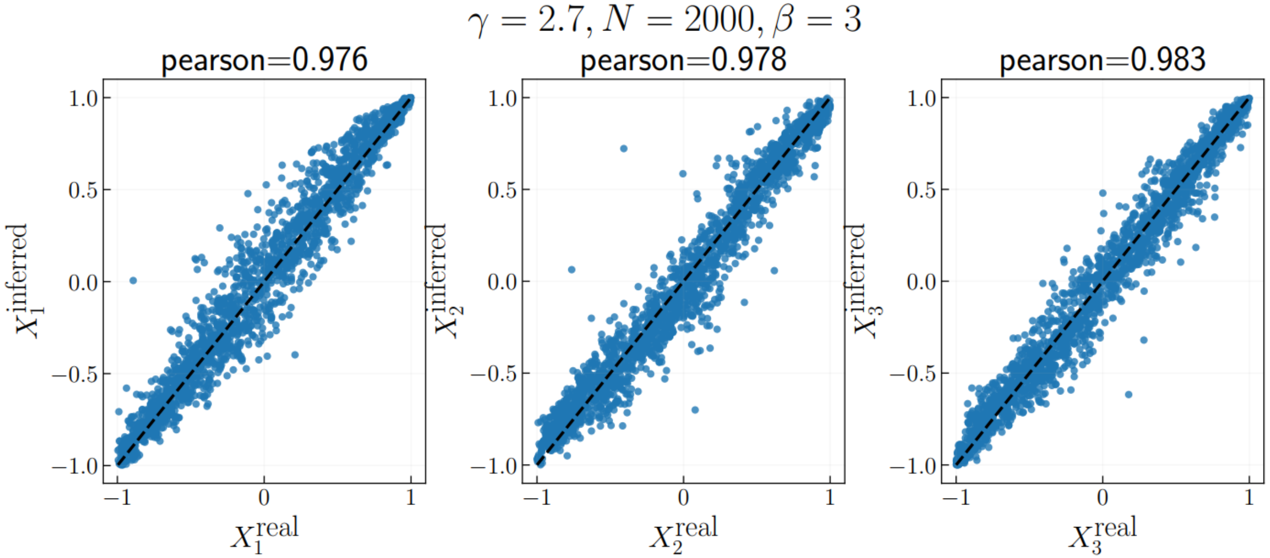
- Likelihood maximization
  - To adjust the positions that maximize the likelihood of the network being generated by the $\mathbb{S}^D$ model
  - Select the most likely candidate positions

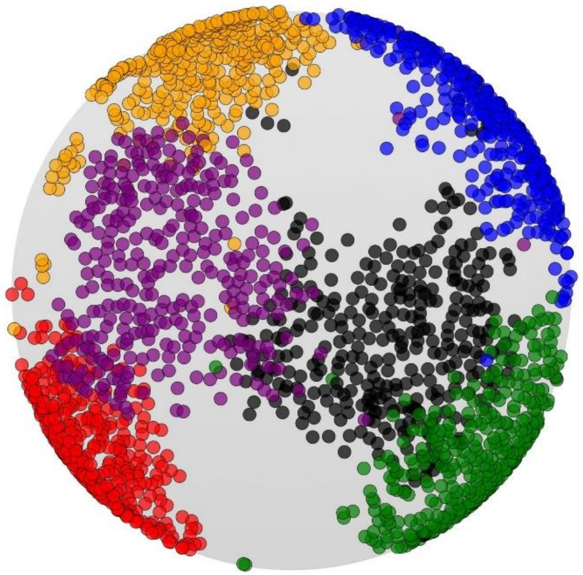
$$\ln \mathcal{L}_i = \sum_{i \neq j} a_{ij} \ln p_{ij} + (1 - a_{ij}) \ln(1 - p_{ij})$$

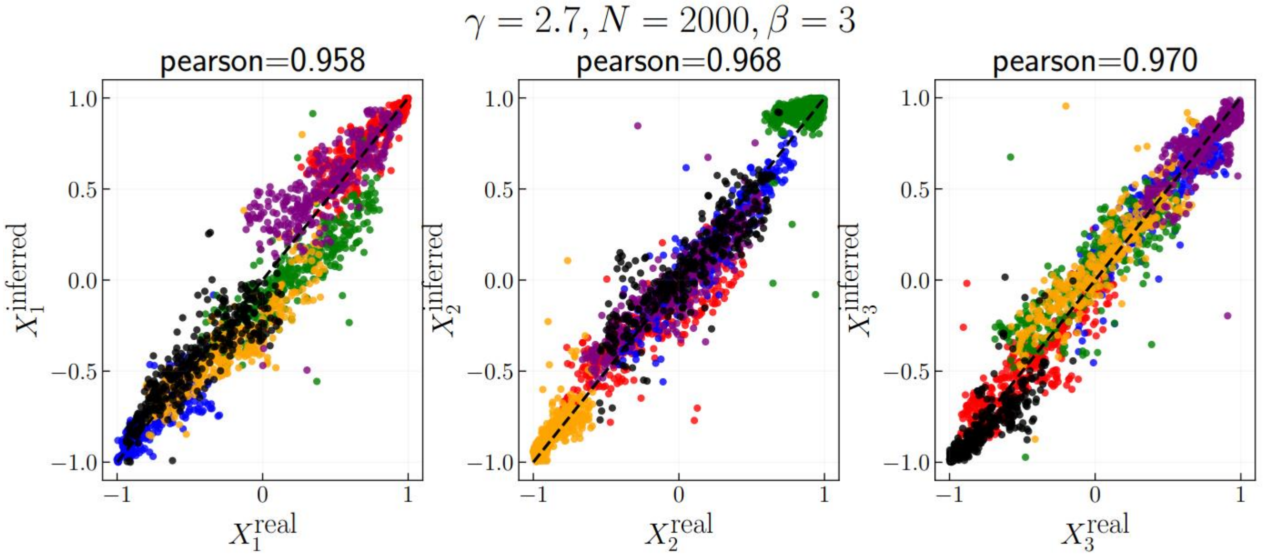
Results on synthetic networks

Validation on synthetic networks

- Comparison between the original positions and the inferred ones in $\mathbb{S}^2$ model
  - (top) without communities
  - (bottom) with 6 communities

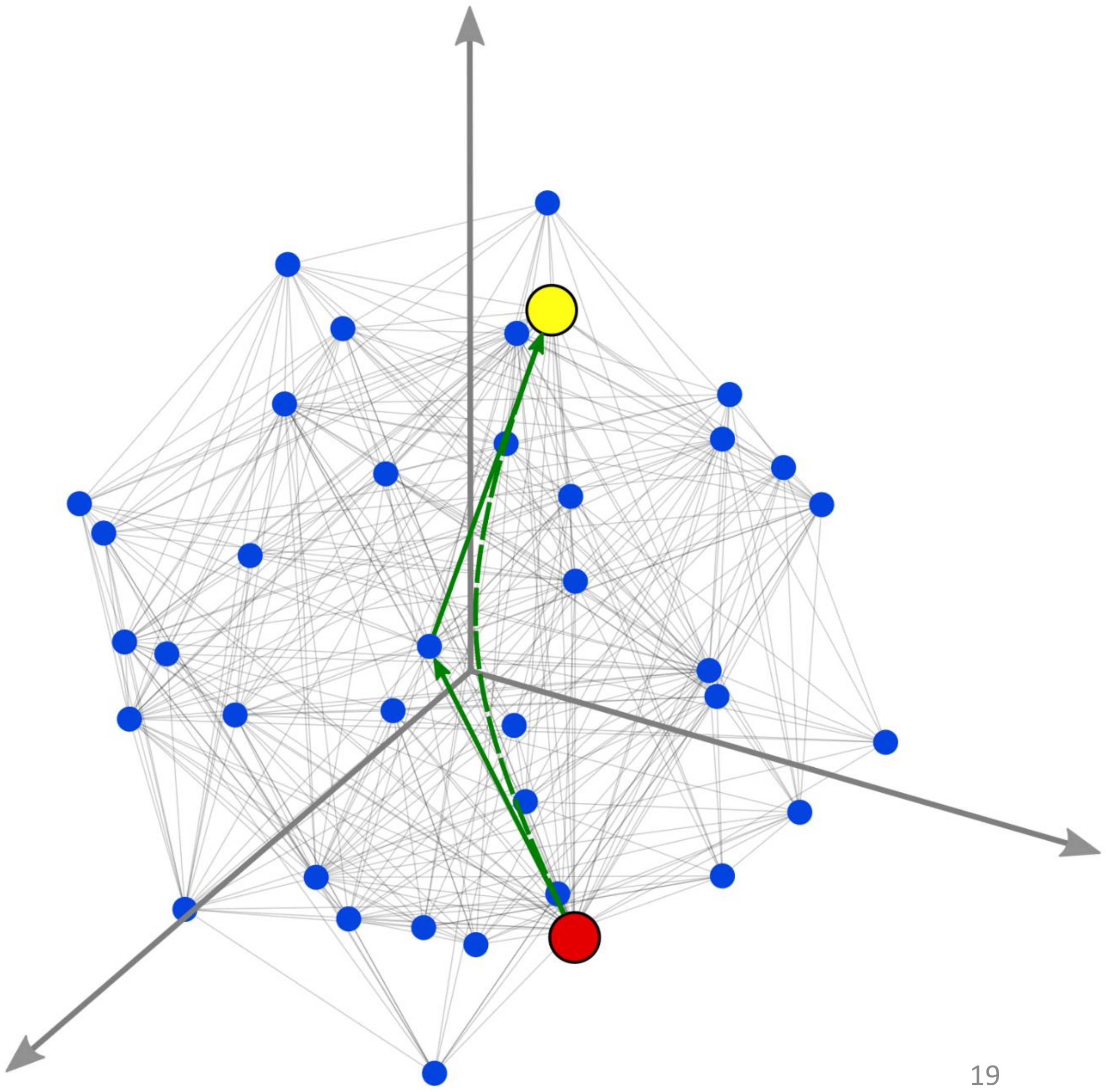




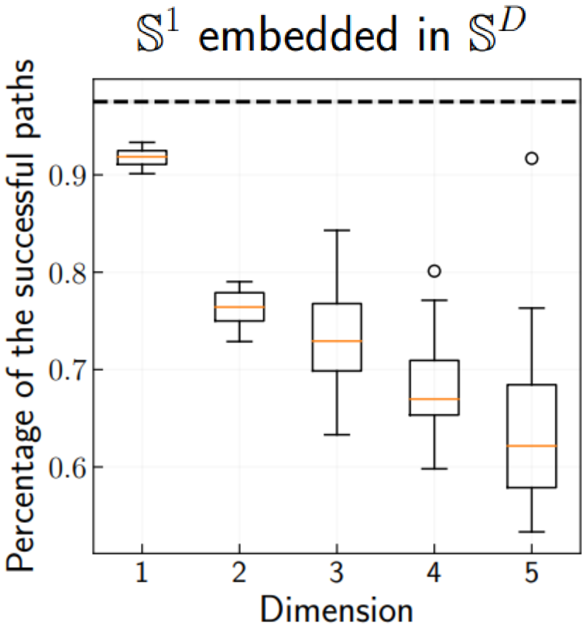


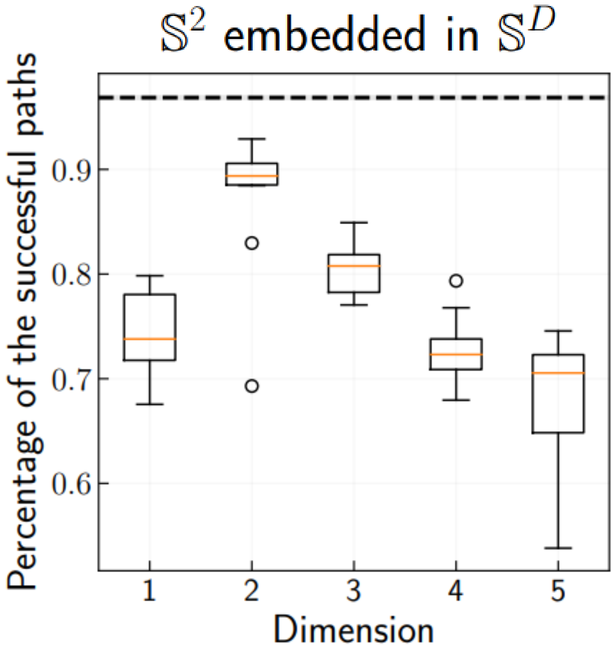
Detecting the hyperbolic dimension of synthetic networks

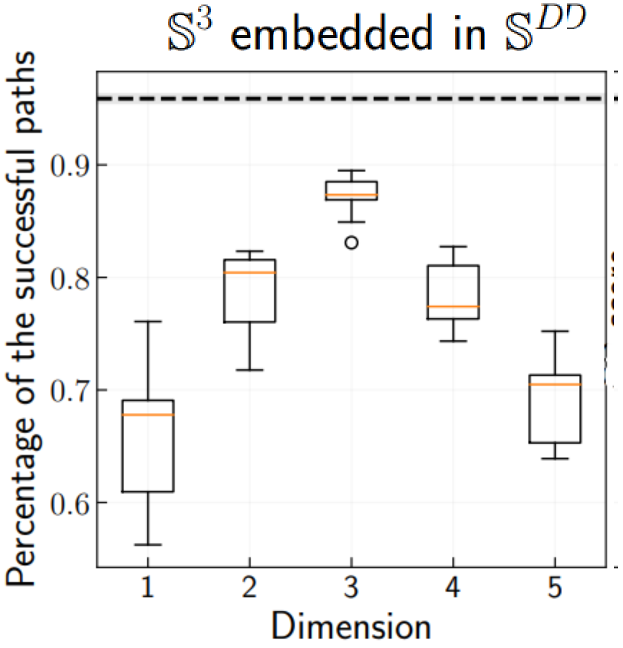
- Navigation (greedy routing)
- Community detection

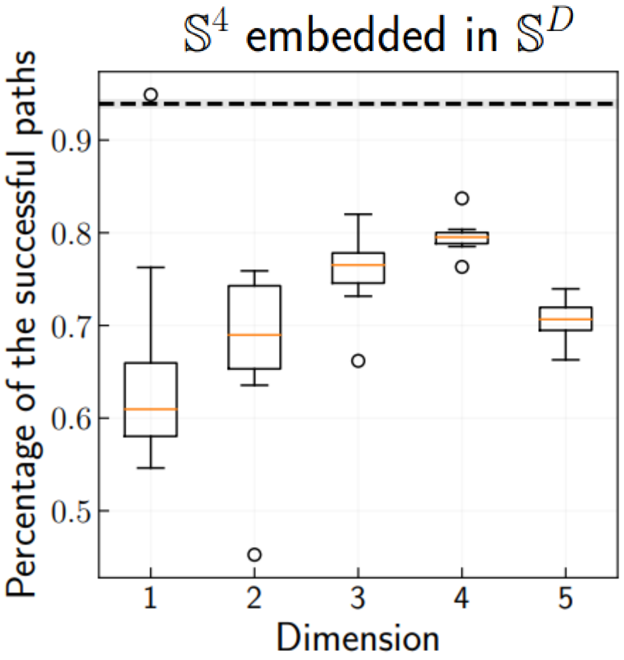


Navigation – results









Synthetic networks with: $N = 2000, \gamma = 2.7, \beta = 2.5D$. Average over 10 realizations.

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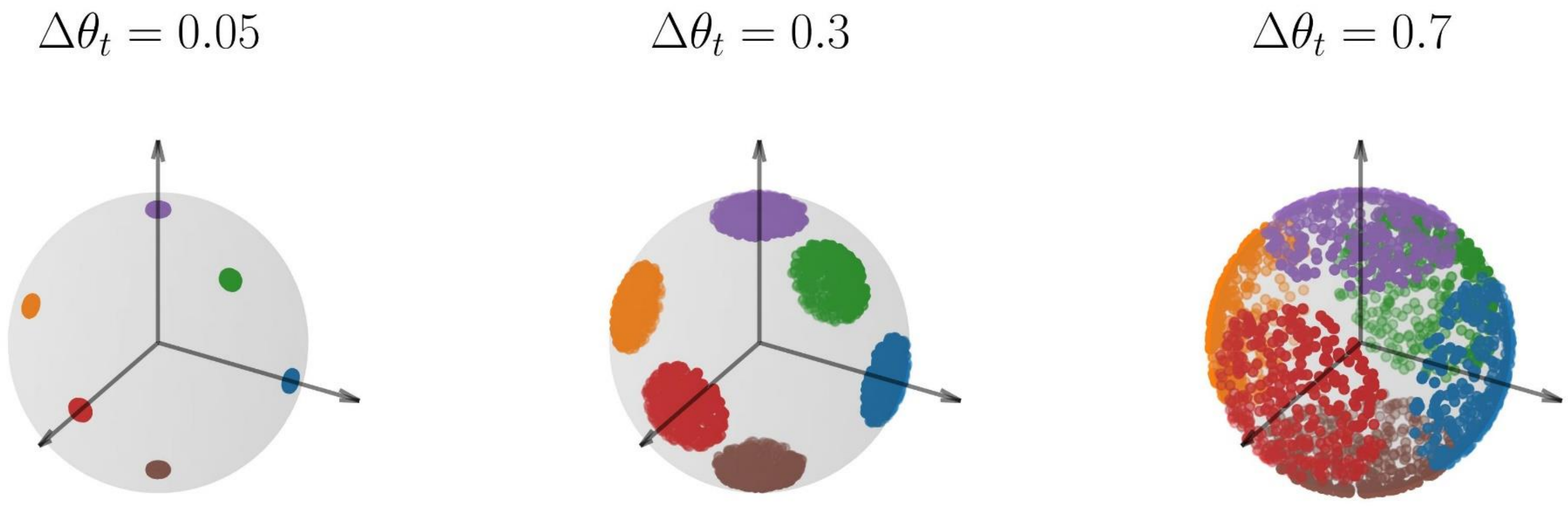
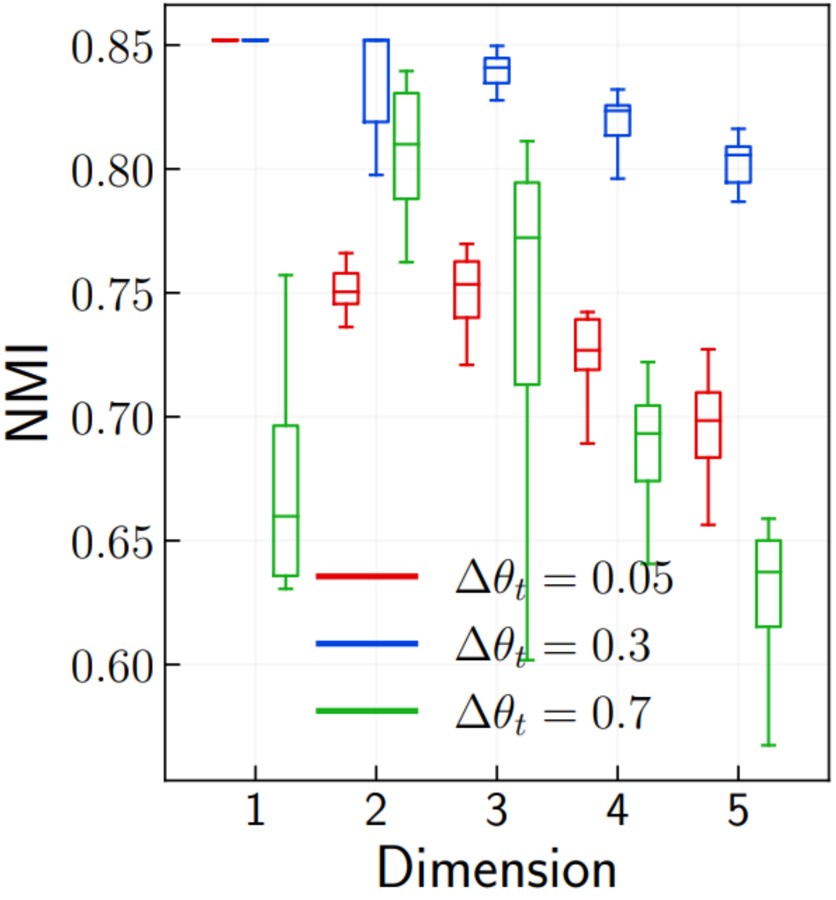
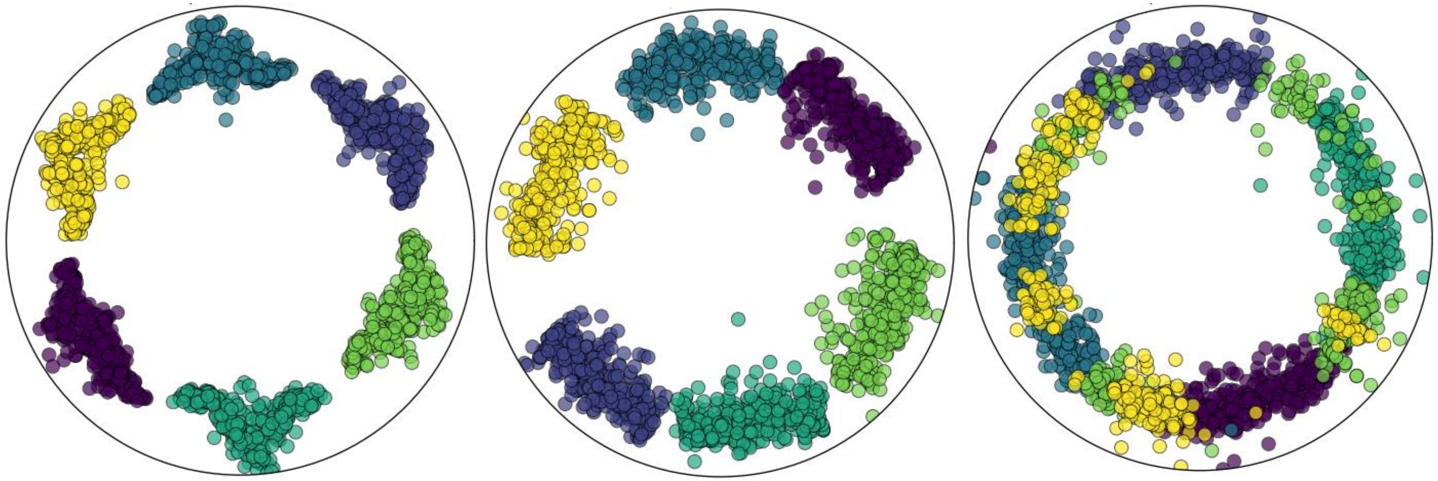


6 communities, nodes within each community are distributed uniformly

Community detection – results

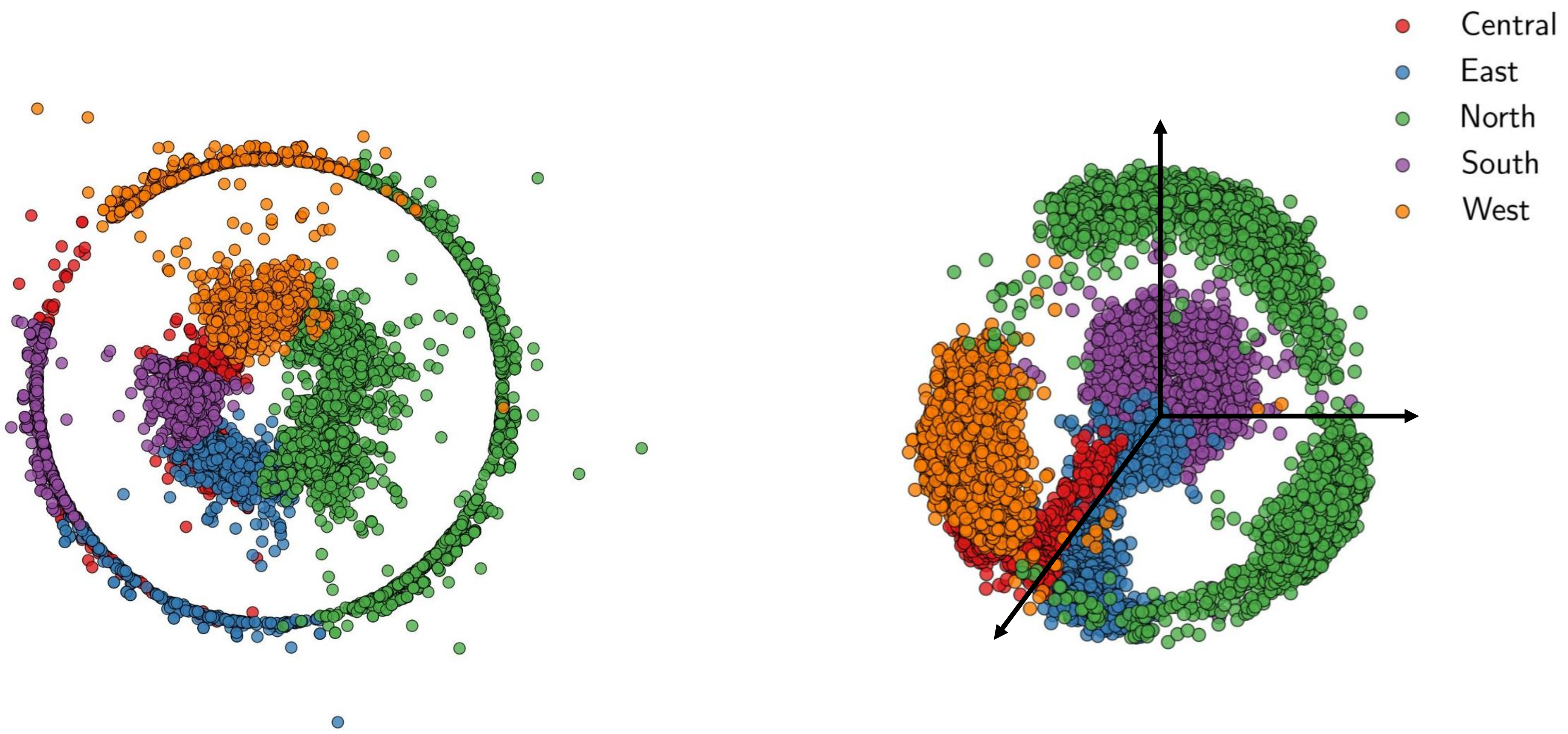


$\Delta\theta_t = 0.05$ $\Delta\theta_t = 0.3$ $\Delta\theta_t = 0.7$



$\beta = 3, \gamma = 2.7, N = 2000$, results averaged over 10 realizations

African urban network – $\mathbb{S}^1$ vs $\mathbb{S}^2$



NMI = 0.61

NMI = 0.73

Conclusions and further steps

Summary

- D-Mercator:
  - a tool to embed networks into multidimensional hyperbolic spaces
  - gives meaningful maps of synthetic networks
- Navigation heavily depends on the dimension of synthetic networks
- Higher dimension helps to unravel the network community structure

In progress

- Determine the intrinsic dimensionality of real networks

Next goal

- What information does the additional dimension give us?
- How does dimension govern the dynamical processes?

Questions?

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Link between $\mathbb{S}^D$ and $\mathbb{H}^{D+1}$model

- Hidden degree of each node κ_i can be mapped to a radial coordinate as

$$r_i = \hat{R} - \frac{2}{D} \ln \frac{\kappa_i}{\kappa_0}$$

- And connection probability becomes

$$p_{ij} = \frac{1}{1 + e^{\frac{\beta}{2}(x_{ij} - \hat{R})}},$$

$$\hat{R} = 2 \ln \left(\frac{2R}{(\mu \kappa_0^2)^{1/D}} \right)$$

- x_{ij} is the hyperbolic distance and can be approximated by

$$x_{ij} = r_i + r_j + 2 \ln \frac{\Delta\theta_{ij}}{2}$$